Effective positivity of Hodge bundles and applications

/w Giulio Codogni & Luca Tasin

/ $\mathbb{C}$

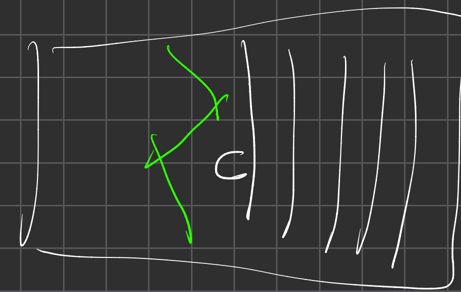
variety:

- can be reducible
- assumed to be projective

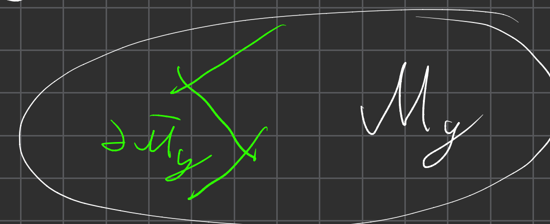
$\boxed{\text{I}}$ Stable varieties

$\boxed{\text{Ia}}$ dim = 1 stable curves

$\overline{\mathcal{U}}_g$



$\boxed{g > 1}$

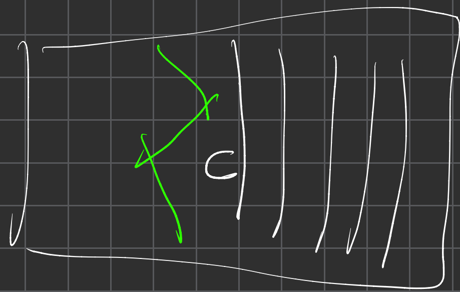


$\overline{\mathcal{M}}_g$ stack

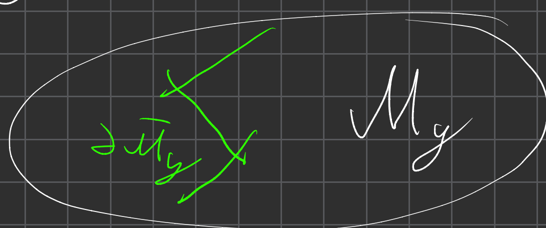
I Stable varieties

Ia $\dim = 1$ stable curves

$\overline{\mathcal{U}}_g$



$\boxed{g > 1}$



$\overline{\mathcal{M}}_g$ stack

C stable curve

1. $\forall$ closed pt is regular / nodal
2. ω_C ample
3. $h^0(\omega_C) = g$

Note: $g > 1 \iff \underbrace{\deg \omega_C}_{\mathrm{vol}(C)} = 2g-2 > 0$

In general: X normal $/k$

K_X $\mathbb{Q}$-Cartier

$\Rightarrow \mathrm{vol}(X) = K_X^n$

C stable curve

① $\forall$ closed pt is regular / nodal

② ω_C ample

[③ $h^0(\omega_C) = g$]

<u>Note</u> $g > 1 \iff \deg \omega_C = 2g-2 > 0$

$\deg \omega_C = \mathrm{vol}(C)$

<u>In general</u>: X normal $/k$

K_X $\mathbb{Q}$-Cartier nef

$\Rightarrow \mathrm{vol}(X) = K_X^n$

$\boxed{\text{I.b}}$ $d = \dim \geq 1$, $0 < v \in \mathbb{Q}$ (or k)

$\left\{\begin{array}{l}\text{semi-canonical} \\ \text{models of} \\ \text{volume}\end{array}\right\} \supseteq \left\{\begin{array}{l}\text{canonical} \\ \text{models} \\ \text{of volume } v\end{array}\right\} \ni \mathrm{Proj}\, R(K_X)$

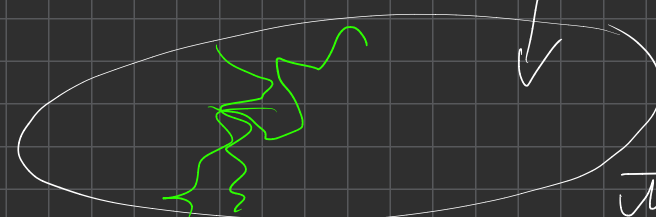
$\|$

$\left\{\begin{array}{l}\text{stable} \\ \text{varieties}\end{array}\right\}$

$\Updownarrow$ $\qquad \uparrow$

$\left\{\begin{array}{l}\text{smooth} \\ \text{var's} \\ \text{of volume} \\ v\end{array}\right\} \ni X$

/ bir eq



$\mathcal{M}_{div}$

$\boxed{\text{I.b}}$ $d = \dim \geq 1$, $0 < v \in \mathbb{Q}$ (or $\mathbb{R}$)

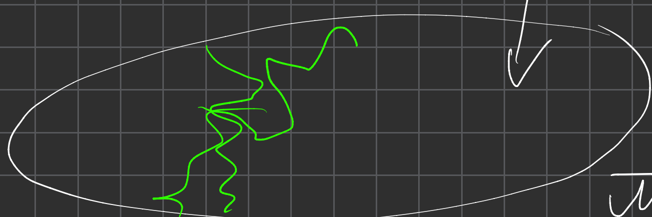
$\left\{ \begin{array}{l} \text{semi-} \\ \text{canonical} \\ \text{models of} \\ \text{volume} \end{array} \right\} \supseteq \left\{ \begin{array}{l} \text{canonical} \\ \text{models} \\ \text{of volume } v \end{array} \right\} \ni \operatorname{Proj} R(K_X)$

$\parallel$ $\Updownarrow$ $\uparrow$

$\left\{ \begin{array}{l} \text{stable} \\ \text{varieties} \end{array} \right\}$ $\left\{ \begin{array}{l} \text{smooth} \\ \text{var's} \\ \text{of volume} \\ v \end{array} \right\} \ni X$

bir eq

$\Updownarrow$



$\overline{\mathcal{M}}_{d,v}$

$\underline{S_6}$, X stable variety:

① semi-log-canonical sings

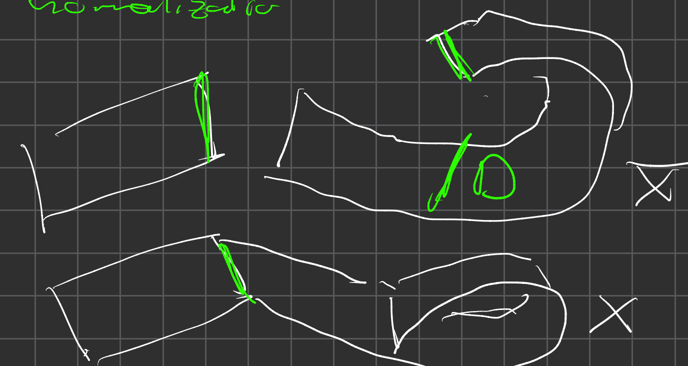
$\boxed{a}$ X demi-normal

[S_2 + regular/nodal in codim 1]

$\boxed{b}$ K_X $\mathbb{Q}$-Cartier

$\boxed{c}$ $(\overline{X}, D)$ log canonical

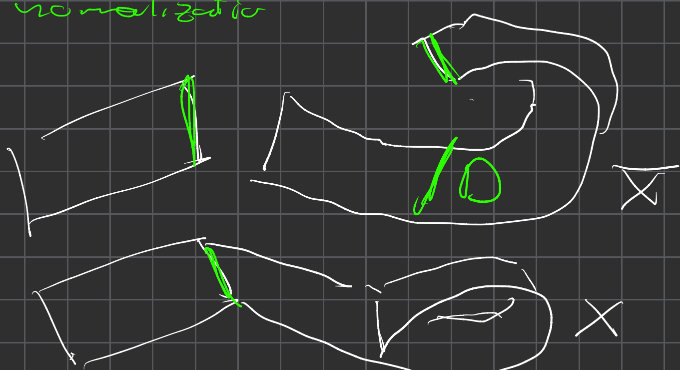
normalization conductor



So, X stable variety:

1. semi log-canonical sings
   - a) X demi-normal
     [S_2 + regular/nodal in codim 1]
   - b) K_X $\mathbb{Q}$-Cartier
   - c) $(\overline{X}, D)$ log canonical

normalization conductor



2. K_X ample

[3. $\dim X = d$
$\mathrm{vol}(X) = v$]

Family of stable var's:

/ T normal ← only this in this talk

- a) flat $f: X \to T$
- b) all fibers are stable
- c) $\boxed{K_{X/T} \ \mathbb{Q}\text{-Cartier}}$

super important

② K_X ample

[③ $\dim X = d$

$\mathrm{vol}(X) = v$]

Family of stable var's:

$/T$ normal ⟵ only this in this talk

a) flat $f : X \to T$

b) all fibers are stable

c) $K_{X/T}$ $\mathbb{Q}$ - Cartier

super important

II Polarization on $\overline{\mathcal{M}}_{d,v}$

$f : (X, \Delta) \to T$, $q \geq 0$

$f_* \mathcal{O}_X (q (K_{X/T} + \Delta))$

$\mathcal{E}_{q,f,\Delta}$

q - th Hodge sheaf

if $\mathcal{E}_{q,f,\Delta}$ comp'ble w base change

$\rightsquigarrow$ $\mathcal{E}_q$ on $\mathcal{M}_{d,v,I}$

coeffs

[even if I is assumed DCC ? $\Longrightarrow$ I finite [Hacon-M^{c}Kernan-Xu]]

$\boxed{\text{II}}$ Polarization on $\overline{\mathcal{M}}_{d,v}$

$f : (X, \Delta) \longrightarrow T$, $\quad q \geq 0$, $\quad q \in \mathbb{Z}$

$f_* \mathcal{O}_X \big(q \, (K_{X/T} + \Delta) \big)$

$\mathcal{E}_{q,f,\Delta}$

$\underline{q\text{-th}}$ Hodge sheaf

If $\mathcal{E}_{q,f,\Delta}$ compatible w/ base change

$\leadsto \mathcal{E}_q$ on $\mathcal{M}_{d,v,I}$

$\uparrow$ coeffs

[even if I is assumed DCC ?
$\Longrightarrow I$ finite [Hacon–McKernan–Xu]]

Thm: det $\mathcal{E}_q$ ample on

1. [Kollár] $\overline{\mathcal{M}}_{2,v}$ for $1 \ll q$
2. [Fujino] $\overline{\mathcal{M}}_{d,v}$ for $1 \ll q$
3. [Kovács-P] $\overline{\mathcal{M}}_{d,v,I}$ for $1 \ll q$

Question
(a) which q?
(b) all q?

Thm: $\exists$ def $\mathcal{E}_q$ ample on

1. [Kollár] $\overline{\mathcal{M}}_{2,v}$ for $1 \ll q$
2. [Fujino] $\overline{\mathcal{M}}_{d,v}$ for $1 \ll q$
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Question (a) which q?

(b) all q?

III Nefness of $\mathcal{E}_{q,f,\Delta}$

$$\begin{tikzcd} (X,\Delta) \arrow[d, "f"] & (X_C, \Delta_C) \arrow[l, hook'] \arrow[d] \\ T & C \arrow[l, hook'] \end{tikzcd}$$

Questions:

0. $\mathcal{E}_{q,f,\Delta}$ flat $/T$?

1. $$\begin{tikzcd} \mathcal{O}_X(q(K_{X/T}+\Delta))|_{X_t} \arrow[d, "??\ \exists\ ??"'] \\ \mathcal{O}_{X_t}(q(K_{X_t}+\Delta_t)) \end{tikzcd}$$

constructed over f semi-smooth + reflexivity

III Nefness of $\mathcal{E}_{q,f,\Delta}$

$$
\begin{tikzcd}
(X,\Delta) \arrow[d, "f"] & (X_C,\Delta_C) \arrow[l, hook'] \arrow[d] \\
T & C \arrow[l, hook']
\end{tikzcd}
$$

$$\mathcal{O}_X(D) = \mathcal{O}_X(\lfloor D \rfloor)$$

Questions:

⓪ $\mathcal{E}_{q,f,\Delta}$ flat $/T$?

① $\mathcal{O}_X(q(K_{X/T}+\Delta))|_{X_t}$

$$\Big\downarrow \; ??\ \cong\ ??$$

constructed over f semi-smooth + reflexivity

$\mathcal{O}_{X_t}(q(K_{X_t}+\Delta_t))$

Note $\Rightarrow$ basechange lC

② $H^i(\mathcal{O}_{X_t}(q(K_{X_t}+\Delta_t)))=0$

$\forall i>0$, $\forall q \geq 2$

③ $(f_C)_* \mathcal{O}_{X_C}(q(K_{X_C/C}+\Delta_C))$

nef $\forall C \subseteq T$?

Thm [Kollár]: ⓪ & ① if $\Delta_q \subseteq \mathbb{Z}$

Thm [Fujino] ② if $\Delta_q \subseteq \mathbb{Z}$

Thm [Codogni–P–Tasin]

Note => basechange lc

(2) $\boxed{H^i(\mathcal{O}_{X_t}(q(K_{X_t}+\Delta_t)))}=0$

$\forall i>0, \ \forall q\geq 2$

(3) $(p_C)_* \mathcal{O}_{X_C}(q(K_{X_C/C}+\Delta_C))$

wf $\forall C\subseteq T$?

Thm: [Kollár]: (0) & (1) if $\Delta_q\subseteq\mathbb{Z}$

Thm [Fujino] (2) if $\Delta_q\subseteq\mathbb{Z}$

Thm [Codogni-P-Tasin]

(3) ✓

Ideas: $\boxed{a}$ we may take

$$\begin{tikzcd}(Y,\Gamma) \arrow[r, "h"] \arrow[rr, bend right, "g"'] & (X,\Delta) \arrow[r, "f"] & T\end{tikzcd}$$

crepant double snc. resolution

$h_*\mathcal{O}_Y(q(K_Y+\frac{1}{q}\lfloor q\Gamma^{>0}\rfloor))$

$\quad\ \ \wr\!\wr$

$\mathcal{O}_X(q(K_{X/T}+\Delta))$

$\boxed{b}$ run $q(K_X+\Delta)$ Cartier proof of [Fujino]

Summary: $\mathcal{E}_{q,f,\Delta}$ wf $\forall q\Delta\subseteq\mathbb{Z}$

Ideas: $\boxed{a}$ we may take

$$\begin{tikzcd} (Y,\Gamma) \arrow[r, "h"] \arrow[rr, bend right, "g"'] & (X,\Delta) \arrow[r, "f"] & T \end{tikzcd}$$

crepant double snc resolution

$$h_* \mathcal{O}_Y\left(q\left(K_Y + \frac{1}{q}\lceil q\Gamma^{>0}\rceil\right)\right)$$
$$\cong$$
$$\mathcal{O}_X(q(K_{X/T}+\Delta))$$

$\boxed{b}$ run $q(K_X+\Delta)$ Cartier

proof of [Fujino]

Summary: $\mathcal{E}_{q,f,\Delta}$ nef $\forall\, q\Delta \subseteq \mathbb{Z}$

$\boxed{IV}$ **Ampleness of $\det \mathcal{E}_q$**

Thm [Codogni–P–Tasin]

$qI \subseteq \mathbb{Z} \rightsquigarrow \det \mathcal{E}_q$ is big on $\forall$ component of $\overline{\mathcal{M}}_{d,v,I}$ containing a klt point

[ample if all pts are klt]

$\boxed{\text{Ideas}}$ ① a general K-V style semipositivity:

- $K_X+\Delta$ $\mathbb{Q}$-Cartier
- gen. fiber klt
- all fibers are red'd
- $L - K_{X/T} - \Delta$ f-big & nef

L $\mathbb{Q}$-Cartier $\mathbb{Z}$-divisor

$$\begin{tikzcd} (X,\Delta) \arrow[d, "f"] \\ T \end{tikzcd}$$

curve

$\boxed{\text{IV}}$ Ampleness of $\det \mathcal{E}_q$

Thm [Codogni–P–Tasin]

$qI \subseteq \mathbb{Z}$ $\rightsquigarrow$ $\det \mathcal{E}_q$ is big on $\forall$ component of $\overline{\mathcal{M}}_{d,v}$ containing a klt point

[ample if all pts are klt]

$\boxed{\text{Ideas}}$ ① a general K-V style semipositivity:

$$\begin{array}{c} (X,\Delta) \ni L \\ \downarrow f \\ T \end{array}$$

L $\mathbb{Q}$-Cartier $\mathbb{Z}$-divisor

T curve

- $K_X+\Delta$ $\mathbb{Q}$-Cartier
- gen. fiber klt
- all fibers are red'd
- $L-K_{X/T}-\Delta$ f-big & nef

$\Rightarrow f_* \mathcal{O}_X(L)$ nef

② ---

$\boxed{\text{V}}$

Thm [---]

same assumptions

$\Rightarrow \mathcal{E}_{2q}$ big

$\boxed{\text{VI}}$ CM line bdl:

$f : (X, \Delta) \to T \rightsquigarrow f_*\left((K_{X/T} + \Delta)^{\dim X/T + 1} \right)$

cycles

$\rightsquigarrow$ extends to a $\mathbb{Q}$-line bdl on $\overline{\mathcal{M}_{n,v,I}}$

$\rightsquigarrow \lambda_{CM}$

Thm [d---]

$\Lambda \subseteq \mathbb{Q} \cap [0,1]$ DCC $\Rightarrow$

$\exists \, \delta(d, \Lambda)$ s.t.

$$\lambda_{CM, f, \Delta}^{\dim T} \leq \delta^{\dim T}$$

$\forall$ stable family $(X, \Delta) \to T$ s.t. $\Delta \in \Lambda$, $\dim X/T = d$ varf max'l & $\exists$ KLT fiber

$\boxed{\text{VII}}$ boundedness of Aut(...)

Thm (...) (w same assump's

$$\mathrm{Aut}(X, \Delta/T) \leq \delta \cdot \mathrm{vol}(K_{X/T} + \Delta)$$